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SOME INVERSE PROBLEMS OF DEFORMATION AND FRACTURE OF PHYSICALLY NONLINEAR INHOMOGENEOUS MEDIA

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The following two types of physically nonlinear inhomogeneous media are considered: linear-elastic plane with nonlinear-elastic elliptic inclusions and linear-viscous plane with elliptic inclusions from a material that possesses nonlinear-creep properties. The problem is to determine infinitely distant loads that produce a required value of the principal shear stress (in the first case) or principal shear-strain rate (in the second case) for two arbitrary inclusions. Conditions for the existence of solutions of these problems for incompressible media under plane strains are obtained.

Key words: *physically nonlinear elliptic inclusions, creep, damage, fracture.*

The present paper is a continuation of the studies [1–3] on modeling the processes of deformation and fracture of physically nonlinear inhomogeneous media. New inverse problems are formulated for linear-elastic and linear-viscous planes with nonlinear (e.g., elastoplastic or nonlinear-viscous) elliptic inclusions in which a homogeneous stress–strain state occurs under the action of loads applied at infinity (provided the distance between the centers of any two inclusions is much greater than their size [1]).

1. Linear-Elastic Plane with Nonlinear-Elastic Inclusions. We consider an isotropic elastic plane S with various physically nonlinear elliptic inclusions (PNEI) located far from one another so that the interaction between the stress–strain states of PNEI can be ignored. We choose two arbitrary inclusions and denote them by S_k^*. Each inclusion is referred to the coordinate system $O_k x_{1k} x_{2k}$ in which the equation of the boundary L_k separating S_k^* from S has the form $x_{1k}^2 a_{0k}^{-2} + x_{2k}^2 b_{0k}^{-2} = 1$ ($a_{0k} \geq b_{0k}$, where $k = 1, 2$). We assume that the inhomogeneous medium is incompressible and undergoes plane strain under the action of remote stresses whose principal values are denoted by N_1 and N_2, and the angle between the first principal axis and the $O_k x_{1k}$ axis is denoted by α_k. The region S obeys Hooke's law [1]

$$4\mu\varepsilon_{22} = -4\mu\varepsilon_{11} = \sigma_{22} - \sigma_{11}, \qquad 2\mu\varepsilon_{12} = \sigma_{12},$$

where σ_{ij} and ε_{ij} are the components of stresses and strains in an arbitrary coordinate system and μ is the shear modulus. (If the latter is replaced by the corresponding Volterra operator, the relations given above describe the plane strain of a linear viscoelastic incompressible medium [1]).

Following [1], we assume that the kth PNEI is isotropic and nonlinear-elastic (or obeys the deformation theory of plasticity). In the coordinate system $O_k x_{1k} x_{2k}$, its constitutive relations have the form

$$\varepsilon_{22k}^* = -\varepsilon_{11k}^* = F_k(\tau_k^*)(\sigma_{22k}^* - \sigma_{11k}^*)/2,$$
$$\varepsilon_{12k}^* = F_k(\tau_k^*)\sigma_{12k}^*, \qquad 2\tau_k^* = [(\sigma_{22k}^* - \sigma_{11k}^*)^2 + 4\sigma_{12k}^{*2}]^{1/2} \qquad (k = 1, 2), \tag{1.1}$$

where $F_k(\tau_k^*) > 0$ is a specified function and τ_k^* is the principal shear stress.

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Since the inclusions do not interact with one another, the stress–strain relations in the kth PNEI and at infinity (provided the rotation at infinity satisfies the condition $\varepsilon^\infty = 0$) take the following form [1]:

$$\mu(m_{0k}\bar{C}_k^* + \bar{D}_k^*) = m_{0k}A_k^* + B_k^* - 2(m_{0k}\Gamma + \Gamma_k'),$$
$$\mu(\bar{C}_k^* + m_{0k}\bar{D}_k^*) = -(A_k^* + m_{0k}B_k^*) + 2\Gamma,$$
$$2A_k^* = \sigma_{11k}^* + \sigma_{22k}^*, \qquad 2B_k^* = \sigma_{22k}^* - \sigma_{11k}^* + 2i\sigma_{12k}^*,$$
$$C_k^* = \varepsilon_{11k}^* + \varepsilon_{22k}^* + 2i\varepsilon_k^*, \qquad D_k^* = \varepsilon_{11k}^* - \varepsilon_{22k}^* + 2i\varepsilon_{12k}^*, \tag{1.2}$$
$$m_{0k} = (a_{0k} - b_{0k})/(a_{0k} + b_{0k}), \qquad 4\Gamma = N_1 + N_2,$$
$$\Gamma_k' = \Gamma_0' \mathrm{e}^{-2i\alpha_k}, \qquad 2\Gamma_0' = N_2 - N_1 \qquad (k = 1, 2).$$

Here ε_k^* is the rotation in S_k^*. We note that the stress–strain state in S_k^* is uniform, i.e., A_k^*, B_k^*, C_k^*, and D_k^* are independent of x_{1k} and x_{2k} $(k = 1, 2)$.

We formulate the problem similar to that considered in [1]: Is it possible to choose the stress–strain state at infinity, i.e., the values of the principal stresses N_1 and N_2 and the angle α_1 (given α_1, the value of α_2 is determined uniquely since the angle $\alpha = \alpha_2 - \alpha_1$ between the axes O_1x_{11} and O_2x_{12} is fixed) in such a manner that the principal shear stress in each PNEI takes a specified value, i.e., the equalities $\tau_k^* = \tau_{0k}$ hold (τ_{0k} are specified values, $k = 1, 2$)?

In contrast to [1] where the principal directions of stresses applied at infinity were specified and the orientation of inclusions (i.e., the angles α_k, $k = 1, 2, \ldots$) was varied, we determine the principal directions for N_1 and N_2 (and their values) for a specified angle α between the centerlines of two PNEI.

We show that the solution of the problem formulated above exists under certain restrictions. Assuming that $B_k^* = \tau_{0k}\mathrm{e}^{i\varphi_k}$, as was done in [1], taking into account the fact that relations (1.1) and (1.2) imply the equalities $C_k^* = 2i\varepsilon_k^*$ and $\bar{D}_k^* = -2F_k(\tau_k^*)B_k^*$, and eliminating A_k^* and ε_k^* from (1.2), we obtain

$$2\Gamma_0' \cos 2\alpha_k = [(1 - m_{0k}^2) + \beta_k(1 + m_{0k}^2)]\tau_{0k}\cos\varphi_k,$$
$$-2\Gamma_0' \sin 2\alpha_k = [(1 + m_{0k}^2) + \beta_k(1 - m_{0k}^2)]\tau_{0k}\sin\varphi_k, \tag{1.3}$$
$$\beta_k = 2\mu F_k(\tau_{0k}) \quad (k = 1, 2), \qquad \alpha_2 = \alpha_1 + \alpha.$$

Relations (1.3) form a system of four equations for Γ_0', α_1, φ_1, and φ_2. In [1], the following necessary conditions for which the system can have solutions were obtained:

$$a_k - |b_k| \le 2|\Gamma_0'| \le a_k + |b_k|,$$
$$a_k \equiv (1 + \beta_k)\tau_{0k} > 0, \qquad b_k \equiv m_{0k}^2(1 - \beta_k)\tau_{0k}, \qquad a_k > |b_k|. \tag{1.4}$$

These conditions follow from the fact that the absolute value of $\cos 2\varphi_k$ that can be obtained from (1.3) cannot exceed unity.

Inequalities (1.4) are satisfied for $k = 1, 2$ if the following inequality holds: $\max\limits_{k=1,2}(a_k - |b_k|) \le \min\limits_{k=1,2}(a_k + |b_k|)$. Analyzing all variants of this inequality, we find that $a_1 - |b_1| \le a_2 + |b_2|$ and $a_2 - |b_2| \le a_1 + |b_1|$, which is equivalent to the condition

$$|a_1 - a_2| \le |b_1| + |b_2|. \tag{1.5}$$

Eliminating the quantities φ_1 and φ_2 from (1.3), we obtain

$$(2\Gamma_0')^{-2} = \frac{\sin^2 2\alpha_1}{(a_1 + b_1)^2} + \frac{\cos^2 2\alpha_1}{(a_1 - b_1)^2} = \frac{\sin^2 2(\alpha_1 + \alpha)}{(a_2 + b_2)^2} + \frac{\cos^2 2(\alpha_1 + \alpha)}{(a_2 - b_2)^2}. \tag{1.6}$$

For $\tan 2\alpha_1$, the last equality yields the quadratic equation

$$(A_2\cos^2 2\alpha + B_2\sin^2 2\alpha - A_1)\tan^2 2\alpha_1 + 2(A_2 - B_2)\sin 2\alpha\cos 2\alpha\tan 2\alpha_1$$
$$+ A_2\sin^2 2\alpha + B_2\cos^2 2\alpha - B_1 = 0, \tag{1.7}$$
$$A_k = (a_k + b_k)^{-2}, \qquad B_k = (a_k - b_k)^{-2} \qquad (k = 1, 2),$$

whose discriminant D is given by

$$D = A\sin^2 2\alpha + B, \quad A = (A_1 - B_1)(A_2 - B_2), \quad B = (A_1 - A_2)(B_2 - B_1). \tag{1.8}$$

We note that the functions A and B given by (1.7) and (1.8) satisfy the following equivalence conditions (since $a_k > |b_k|$, where $k = 1, 2$):

$$A > 0 \quad (< 0) \quad \Longleftrightarrow \quad b_1 b_2 > 0 \quad (< 0); \tag{1.9}$$

$$B \geq 0 \quad (\leq 0) \quad \Longleftrightarrow \quad |b_1 - b_2| - |a_1 - a_2| \geq 0 \quad (\leq 0); \tag{1.10}$$

$$B + A \geq 0 \quad (\leq 0) \quad \Longleftrightarrow \quad |b_1 + b_2| - |a_1 - a_2| \geq 0 \quad (\leq 0). \tag{1.11}$$

For $\tan 2\alpha_1$, Eq. (1.7) has real roots if $D \geq 0$. This condition is satisfied in the following cases:

1) For all values of $\sin 2\alpha$ if $A > 0$ and $B \geq 0$, i.e., if according to (1.9) and (1.10)

$$b_1 b_2 > 0, \qquad |a_1 - a_2| \leq |b_1 - b_2| = ||b_1| - |b_2||$$

[condition (1.5) holds since $||b_1| - |b_2|| \leq |b_1| + |b_2|$], or if $A < 0$, $B > 0$, and $A + B \geq 0$ (since $D \geq A + B$), which, by virtue of (1.9)–(1.11), is equivalent to the inequalities

$$b_1 b_2 < 0, \qquad |a_1 - a_2| \leq |b_1 + b_2| = ||b_1| - |b_2||;$$

2) If the inequality

$$\sin^2 2\alpha \geq -\frac{B}{A} = \frac{[(a_2 + b_2)^2 - (a_1 + b_1)^2][(a_2 - b_2)^2 - (a_1 - b_1)^2]}{16 a_1 b_1 a_2 b_2}$$

holds for $A > 0$ and $B < 0$, i.e., $b_1 b_2 > 0$ and $|a_1 - a_2| > |b_1 - b_2| = ||b_1| - |b_2||$. This is possible if $-B/A \leq 1$, i.e., $A + B \geq 0$ or $|a_1 - a_2| \leq |b_1 + b_2| = |b_1| + |b_2|$ [see (1.11)], which coincides with (1.5);

3) If the inequalities $\sin^2 2\alpha < -B/A < 1$ are satisfied for $A < 0$, $B > 0$, and $A + B < 0$, i.e., $b_1 b_2 < 0$ and

$$||b_1| - |b_2|| = |b_1 + b_2| < |a_1 - a_2| < |b_1 - b_2| = |b_1| + |b_2|.$$

Thus, if

$$|a_1 - a_2| \leq ||b_1| - |b_2||, \tag{1.12}$$

then $D \geq 0$ regardless of $\operatorname{sign}(b_1 b_2)$.

If

$$||b_1| - |b_2|| < |a_1 - a_2| < |b_1| + |b_2|, \tag{1.13}$$

then $D \geq 0$ for $\sin^2 2\alpha \geq -B/A$ (for $A > 0$) or for $\sin^2 2\alpha \leq -B/A$ (for $A < 0$).

Once the values of α_1 are found, the quantity $|\Gamma_0'|$ and the angles φ_k $(k = 1, 2)$ are determined from (1.6) and (1.3), respectively. As is shown in [1], if the inequality

$$[\tau \beta_k(\tau)]' \geq 0 \qquad (k = 1, 2) \tag{1.14}$$

(the prime denotes differentiation with respect to τ), which follows from the stability condition for the constitutive equations (1.1), holds and the values of Γ_0' and α_k are specified, then τ_k and φ_k $(-\pi \leq \varphi_k \leq \pi)$ are determined uniquely, i.e., only the values of $\tau_k^* = \tau_{0k}$ $(k = 1, 2)$ correspond to the values of Γ_0' and α_k determined by solving the problem formulated above.

We consider some particular cases where the solution of Eq. (1.7) for α_1 exists. Let both PNEI have the same properties, i.e., $F_k = F$ $(k = 1, 2)$ in (1.1), and it is required that the principal shear stresses occurring in them are equal: $\tau_{01} = \tau_{02}$. In this case, we have $a_1 = a_2$ and inequality (1.12) is satisfied for both $m_{01} = m_{02}$ and $m_{01} \neq m_{02}$; hence, the solution exists for all α, i.e., for all orientations of the inclusions relative to each other.

If $\tau_{01} \neq \tau_{02}$ and $m_{01} = m_{02} = m_0$, inequality (1.12) fails. Indeed, if inequality (1.14) is satisfied, both functions $[1 + \beta(\tau) \pm m_0^2(1 - \beta(\tau))]\tau$, where $\beta(\tau) = 2\mu F(\tau)$, are increasing functions and, hence, for $\tau_{01} > \tau_{02}$, we obtain $a_1 > a_2$, $a_1 \pm b_1 > a_2 \pm b_2$. This implies that $a_1 - a_2 > b_2 - b_1$, $a_1 - a_2 > b_1 - b_2$, i.e., $|a_1 - a_2| = a_1 - a_2 > |b_1 - b_2| \geq ||b_1| - |b_2||$. In a similar manner, we obtain $|a_1 - a_2| > ||b_1| - |b_2||$ for $\tau_{02} > \tau_{01}$. Thus, in this case, the solution for α_1 exists if the second inequality (1.13) and the above-mentioned restrictions imposed on the value of α are satisfied.

2. Linear-Viscous Plane with Nonlinear-Viscous Inclusions. We formulate the problem similar to that considered above for an isotropic linear-viscous plane with nonlinear-viscous elliptic inclusions (NVEI) assuming that the distance between the centers of the inclusions is much greater than their size. We ignore the elastic strains of this inhomogeneous medium and assume that the medium is incompressible and subjected to plain strain. In the viscous region S, we obtain the following relations similar to Hooke's law relations (see Sec. 1):

$$4\mu\eta_{22} = -4\mu\eta_{11} = \sigma_{22} - \sigma_{11}, \qquad 2\mu\eta_{12} = \sigma_{12}.$$

Here η_{ij} are the strain-rate components and μ is the viscosity coefficient.

We write the constitutive equations for the kth NVEI in the coordinate system $O_k x_{1k} x_{2k}$ determined by its centerlines in the form [2, 3]

$$\eta^*_{22k} = -\eta^*_{11k} = \frac{H^*_k}{2\tau^*_k}\,\frac{\sigma^*_{22k} - \sigma^*_{11k}}{2}, \qquad \eta^*_{12k} = \frac{H^*_k}{2\tau^*_k}\,\sigma^*_{12k},$$

$$H^*_k = \frac{B_{1k}\tau_k^{*n_k}}{(1-\omega_k)^{q_k}}, \qquad \dot{\omega}_k = \frac{B_{2k}\tau_k^{*p_k}}{(1-\omega_k)^{q_k}}, \tag{2.1}$$

$$H^*_k = [(\eta^*_{22k} - \eta^*_{11k})^2 + 4\eta^{*2}_{12k}]^{1/2} \qquad (k = 1, 2).$$

Here η^*_{ijk} are the strain-rate components of the inclusion, H^*_k is the principal shear-strain rate, ω_k $(0 \le \omega_k \le 1)$ is the damage parameter ($\omega_k = 0$ for the undeformed state and $\omega_k = 1$ at the moment of fracture), B_{1k}, B_{2k}, n_k, p_k, and q_k are positive constants, the dot denotes differentiation with respect to time t, and the remaining notation is the same as in Sec. 1.

Relations (2.1) can be inverted [2]:

$$\frac{\sigma^*_{22k} - \sigma^*_{11k}}{2} = \frac{2\tau^*_k}{H^*_k}\,\eta^*_{22k}, \qquad \sigma^*_{12k} = \frac{2\tau^*_k}{H^*_k}\,\eta^*_{12k}, \qquad \dot{\omega}_k = B_{0k} H_k^{*\gamma_k}(1-\omega_k)^{æ_k},$$

$$B_{0k} = B_{2k} B_{1k}^{-\gamma_k}, \qquad \gamma_k = p_k/n_k, \qquad æ_k = q_k(\gamma_k - 1). \tag{2.2}$$

Relations (2.1) and (2.2) describe the processes of isothermal creep deformation and fracture of brittle and viscous materials.

The stress–strain relations in the kth NVEI and at infinity are given by (1.2), where ε^*_{ijk} should be replaced by η^*_{ijk} $(i, j = 1, 2)$ and ε^*_k by $\dot{\varepsilon}^*_k$ and the quantity μ should be understood as the viscosity coefficient.

The problem is formulated as follows: Is it possible to choose the stress–strain state at infinity, i.e., the stresses N_1 and N_2 and the angle α_1 such that the principal shear-strain rate in each inclusion takes the required value, i.e., $H^*_k = H_{0k}(t)$? Here $H_{0k}(t)$ $(k = 1, 2)$ are specified functions of time. For $t = 0$, the NVEI are not deformed and, hence, $\omega_k|_{t=0} = 0$ $(k = 1, 2)$.

Relation (1.2) with modifications considered above and relation (2.2) yield the equalities $|\bar{D}^*_k| = H^*_k$, $C^*_k = 2i\dot{\varepsilon}^*_k$, and $B^*_k = -\tau^*_k H_k^{*-1}\bar{D}^*_k$. Setting $\bar{D}^*_k = H_{0k}\mathrm{e}^{i\varphi_k}$, by analogy with (1.3), we obtain [3]

$$-2\Gamma'_0 \cos 2\alpha_k = [(1 - m^2_{0k})\tau^*_k + \mu(1 + m^2_{0k})H_{0k}]\cos\varphi_k,$$

$$2\Gamma'_0 \sin 2\alpha_k = [(1 + m^2_{0k})\tau^*_k + \mu(1 - m^2_{0k})H_{0k}]\sin\varphi_k, \tag{2.3}$$

$$\tau^*_k = [B_{1k}^{-1} H_{0k}(1-\omega_k)^{q_k}]^{1/n_k} \quad (k = 1, 2), \qquad \alpha_2 = \alpha_1 + \alpha.$$

System (2.3) for the unknowns Γ'_0, α_1, φ_1, and φ_2 has a solution only if inequalities (1.4) are satisfied with allowance for the relations

$$a_k = \tau^*_k + \mu H_{0k}, \qquad b_k = m^2_{0k}(\tau^*_k - \mu H_{0k}). \tag{2.4}$$

For the specified functions $H_{0k}(t)$, the quantities τ^*_k in (2.3) and (2.4) are found after determination of the damage parameters ω_k by integrating the third group of equations (2.2) under the initial conditions $\omega_k|_{t=0} = 0$ $(k = 1, 2)$.

Proceeding as was done in Sec. 1, we obtain formulas (1.5)–(1.13), in which a_k and b_k are determined according to (2.4). In particular, if (1.12) holds, the solution for α_1 exists for any angle α between the symmetry lines of the inclusions; if inequalities (1.13) hold, the restrictions mentioned above are imposed on α.

In [3], we show that, if the values of $\Gamma'_0 = \Gamma'_0(t)$ and $\alpha_k = \alpha_k(t)$ are specified, the functions $H^*_k = H^*_k(t)$ and $\varphi_k = \varphi_k(t)$ $(-\pi \le \varphi_k \le \pi)$ are uniquely determined from (2.2) and (2.3), i.e., only the values of $H^*_k = H_{0k}$ correspond to the values of Γ'_0 and α_k determined by solving the problem formulated in this section.

The problem considered can also be interpreted as follows: Is it possible to choose stresses at infinity such that $H_k^* = H_{0k}(t)$ and, hence, the kth inclusion fails in a specified period t_k^* $(k = 1, 2)$? Indeed, the quantities mentioned above are related by the equality [2, 3]

$$(1 - æ_k) B_{0k} \int\limits_0^{t_k^*} H_k^{*\gamma_k}\, dt = 1 \qquad (æ_k < 1),$$

which follows from the third group of equations (2.2) and conditions $\omega_k|_{t=0} = 0$ and $\omega_k|_{t=t_k^*} = 1$ $(k = 1, 2)$.

In a particular case where $H_k^* = \text{const}$, we obtain

$$t_k^{*-1} = (1 - æ_k) B_{0k} H_k^{*\gamma_k}.$$

Thus, one can formulate the problem of optimal fracture of two NVEI, i.e., fracture in a specified time t_k^*.

In conclusion, the following remark should be made. In [3], the problem similar to that considered above was formulated for k inclusions whose orientation relative to each other was fixed; the case $k > 2$ was admissible. In the formulation proposed in the present paper, however, this is impossible since, as was shown above, setting of H_{0k} and α for four unknown quantities Γ_0, α_1, and φ_k $(k = 1, 2)$ yields a closed system of four equations consisting of first two equalities (2.3), where $\alpha_2 = \alpha_1 + \alpha$ and $k = 1, 2$. If a couple of equations corresponding, e.g., to the case $k = 3$ are added to the system, the latter becomes overdetermined [six equations for five unknowns Γ_0, α_1, and φ_k $(k = 1, 2, 3)$ since α_2 and α_3 are expressed in terms of α_1]. This imposes restrictions on the orientation of other inclusions, i.e., on the angles α_k for $k \geq 3$. One can easily show that, in the general case, $2k$ equations can be written for $2 + k$ unknown functions.

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